Asymptotic Analysis of the Magnetorotational Instability

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Motivation

- Magnetic field-induced instabilities can transport angular momentum in astrophysical accretion disks outwards, thereby permitting accretion
- Magnetorotational instability has several appealing properties (Balbus and Hawley 1991, 1998)
  - It is a linear instability
  - It is triggered by weak poloidal magnetic field
  - It is axisymmetric
  - It occurs in Rayleigh-stable regime when the angular velocity decreases radially
  - It grows on a dynamical timescale
  - It is fundamentally a local instability
- Efficiency of angular momentum transport depends on the saturation of the MRI
- Central question: how does the MRI saturate?
  This is a **nonlinear** problem!
- Is the saturated state perhaps a dynamo? Lesur and Ogilvie (2008)

Ideal MRI

The basic equations are

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p - \frac{1}{2\mu_0}\nabla B^2 + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B},$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{u},$$

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = 0.$$

These equations have a basic axisymmetric solution of the form

$$\mathbf{u_0} = [0, V(r), 0], \qquad \mathbf{B_0} = [0, B_\phi(r), B_z(r)]$$

in (r, ϕ, z) coordinates.

Numerical simulations: shearing box geometry

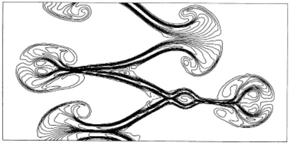


FIG. 5b

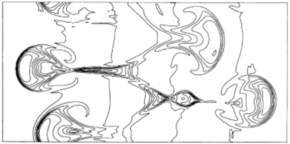
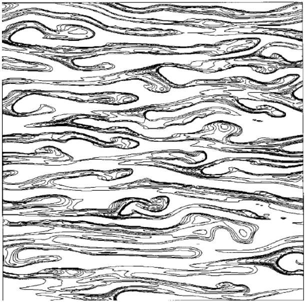


FIG. 5c



- Balbus-Hawley 1991a,b: Thin sheets of matter moving radially inwards and outwards
- X-points suggest reconnection process important to saturation
- Goodman & Xu 1994, Pessah & Goodman 2009: shear instabilities of the interpenetrating sheets
- Sano et al 1998: whether saturation occurs depends on the Elsasser number $\Lambda \equiv v_A^2/\eta\Omega$

Formulation of a Model Problem: Knobloch & Julien 2005

- Shearing box approximation at r^* with local angular velocity $\Omega^*(r^*)\hat{\mathbf{z}}$:
- Straight channel: $-L^*/2 \le x^* \le L^*/2$, $-\infty < y^* < \infty$, $-\infty < z^* < \infty$
- Linear shear: $\mathbf{U_0}^* = (0, \sigma^* x^*, 0)$
- Constant B-Field: $\mathbf{B_0}^* = (0, B^*_{tor}, B^*_{pol})$
- Perturb: $\mathbf{u} \equiv (u, v, w) = (-\psi_z, v, \psi_x)$, $\mathbf{b} \equiv (a, b, c) = (-\phi_z, b, \phi_x)$

  Axisymmetric Equations

$$\nabla^2\psi_t + 2\Omega v_z + J(\psi, \nabla^2\psi) = v_A^2\nabla^2\phi_z + v_A^2 J(\phi, \nabla^2\phi) + \nu\nabla^4\psi, \quad (1)$$

$$v_t - (2\Omega + \sigma)\psi_z + J(\psi, v) = v_A^2 b_z + v_A^2 J(\phi, b) + \nu\nabla^2 v, \quad (2)$$

$$\phi_t + J(\psi, \phi) = \psi_z + \eta\nabla^2\phi, \quad (3)$$

$$b_t + J(\psi, b) = v_z - \sigma\phi_z + J(\phi, v) + \eta\nabla^2 b, \quad (4)$$

where $J(f, g) \equiv f_x g_z - f_z g_x$.

- $v_A \equiv B^*_{pol}/\sqrt{\mu_0\rho^*}U^*$, Ω, ν, η are the *dimensionless* Alfvén speed, rotation rate, kinematic viscosity and ohmic diffusivity

Linear Theory

- Linearization about the trivial state $\psi = v = \phi = b = 0$:
- Perturbation $\exp[\lambda t + ikx + inz]$, $p = k^2 + n^2 \Rightarrow$ dispersion relation

$$p[(\lambda+\nu p)(\lambda+\eta p)+v_A^2 n^2]^2+2\Omega n^2[(\lambda+\eta p)^2(2\Omega+\sigma)+\sigma v_A^2 n^2] = 0. \quad (5)$$

- Conventional view of MRI: positive growth rate λ achieved for sufficiently large vertical wavenumbers n whenever $\sigma < 0$, $v_A \neq 0$, provided only that ν and η are sufficiently small
  - For $\nu = \eta = 0$

$$\lambda^2 = -\frac{v_A^2 n^2 \sigma}{2\Omega + \sigma} + O(v_A^4 n^4). \quad (6)$$

  - For $\lambda = 0$ threshold for instability exists. For small ν, η critical Elsasser number

$$\Lambda_c \equiv v_A^2/\Omega\eta = -\eta\left(\frac{2\Omega + \sigma}{\Omega\sigma}\right)\frac{p^2}{n^2} + O(\nu, \eta)^3. \quad (7)$$

  - Reconnection effects described by finite η are more important for stabilizing the system against the MRI than viscosity.

Scaling Assumptions

- Traditional approach to nonlinear saturation: weakly nonlinear theory with $(\Lambda - \Lambda_c)/\Lambda_c \ll 1$ (eg. Umurhan & Regev 2007)
- Our approach: strongly nonlinear theory
  - shear is the dominant source of energy for the MRI
  - MRI itself requires the presence of a (weaker) vertical magnetic field
  - dissipative effects are weaker still but cannot be ignored since they are ultimately responsible for the saturation of the instability
- Hence scaling:
  - rapid rotation, strong shear: $(\Omega, \sigma) = \epsilon^{-1}(\hat{\Omega}, \hat{\sigma})$
  - magnetic field: $v_A = 1$ i.e. , $U^* = v_A^* \equiv B_{pol}^*/\sqrt{\mu_0 \rho^*}$
  - weak dissipative processes: $(\nu, \eta) = \epsilon(\hat{\nu}, \hat{\eta})$
  - thin fingers, strong growth: $\partial_x \to \partial_x, \quad \partial_z \to \epsilon^{-1}\partial_z, \quad \partial_t \to \epsilon^{-1}\partial_t$
- In the following we take $\epsilon \ll 1$, or equivalently $Rm \gg S \gg \max(1, Pm)$, while $\Lambda = O(1)$. Here $Rm = |\sigma^*|L^{*2}/\eta^* = O(\epsilon^{-2})$, $Pm = \nu^*/\eta^* = O(1)$, $S \equiv v_A^* L^*/\eta^* = O(\epsilon^{-1})$ are the magnetic Reynolds, magnetic Prandtl and Lundquist numbers.

Scaled Equations

- In parallel with the above assumptions we need to make further assumptions about the relative magnitude of the various fields:
- we find $(\psi, \phi) \to \epsilon(\psi, \phi)$, $(v, b) \to \epsilon^{-1}(v, b)$ leads to a self-consistent set of reduced pdes
- scaled pdes:

$$\epsilon \frac{D}{Dt}\left(\partial_x^2 + \epsilon^{-2}\partial_z^2\right)\psi + 2\epsilon^{-3}\hat{\Omega}v_z = v_A^2\left(\partial_x^2 + \epsilon^{-2}\partial_z^2\right)\phi_z + \epsilon v_A^2 J\left(\phi, \left(\partial_x^2 + \epsilon^{-2}\partial_z^2\right)\phi\right) + \epsilon^2\hat{\nu}\left(\partial_x^2 + \epsilon^{-2}\partial_z^2\right)^2\psi \tag{8}$$

$$\epsilon^{-1}\frac{D}{Dt}v - \epsilon^{-1}(2\hat{\Omega} + \hat{\sigma})\psi_z = \epsilon^{-2}v_A^2 b_z + \epsilon^{-1}v_A^2 J(\phi, b) + \hat{\nu}(\partial_x^2 + \epsilon^{-2}\partial_z^2)v \tag{9}$$

$$\epsilon \frac{D}{Dt}\phi = \psi_z + \epsilon^2\hat{\eta}(\partial_x^2 + \epsilon^{-2}\partial_z^2)\phi \tag{10}$$

$$\epsilon^{-1}\frac{D}{Dt}b = \epsilon^{-2}v_z - \epsilon^{-1}\hat{\sigma}\phi_z + \epsilon^{-1}J(\phi, v) + \hat{\eta}(\partial_x^2 + \epsilon^{-2}\partial_z^2)b, \tag{11}$$

where $D/Dt = \partial_t + J[\psi, \bullet]$.

Derivation of Reduced PDEs

- To solve the scaled equations we suppose $\psi(x,z,t) = \psi_0(x,z,t) + \epsilon\psi_1(x,z,t) + \ldots$, etc.
- $$v_0 = V(x), \qquad b_0 = B(x). \tag{12}$$
- From Eqs for azimuthal fields v, b at $O(\epsilon^{-1})$ and poloidal fields ψ, ϕ at $O(\epsilon^{-2})$, $O(1)$

$$\psi_{0zzt} + 2\widehat{\Omega} v_{1z} = v_A^2 \phi_{0zzz} + \widehat{\nu}\psi_{0zzzz} \tag{13}$$

$$v_{1t} - (2\widehat{\Omega} + \widehat{\sigma} + V'(x))\psi_{0z} = v_A^2 b_{1z} - v_A^2 B'(x)\phi_{0z} + \widehat{\nu} v_{1zz} \tag{14}$$

$$\phi_{0t} = \psi_{0z} + \widehat{\eta}\phi_{0zz} \tag{15}$$

$$b_{1t} - B'(x)\psi_{0z} = v_{1z} - (\widehat{\sigma} + V'(x))\phi_{0z} + \widehat{\eta} b_{1zz} \tag{16}$$

- Closure requires determination of $V'(x), B'(x)$.
  - Averaging Eqs for azimuthal fields v, b at $O(1)$ in z, t and integrating gives

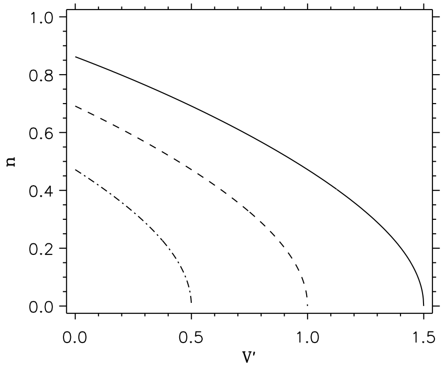
$$\widehat{\nu} V'(x) = \overline{\psi_0 v_{1z}} - v_A^2 \overline{\phi_0 b_{1z}} + C_1 \tag{17}$$

$$\widehat{\eta} B'(x) = \overline{\psi_0 b_{1z}} - \overline{\phi_0 v_{1z}} + C_2 \tag{18}$$

Nonlinear Dispersion Relation

- For each wavenumber n the dispersion relation determines V'

$$2\hat{\Omega}[(v_A^2 + \hat{\eta}^2 n^2)V' + (2\hat{\Omega} + \hat{\sigma})\hat{\eta}^2 n^2 + \hat{\sigma} v_A^2] + n^2(v_A^2 + \hat{\nu}\hat{\eta} n^2)^2 = 0$$



Parameters: $\widehat{\Omega} = 1, v_A = 1, \widehat{\nu} = \widehat{\eta} = 1$, and $\widehat{\sigma} = -1.5, -1, -0.5$ (solid, dashed, dashed-dot).

The decrease in n with increasing V' indicates coarsening as the MRI saturates.

Single-Mode Solutions: Closure

- Closure requires the determination of V', B' as a function of Ψ. Since

$$\psi_0 = \frac{1}{2}(\Psi(x)\, e^{inz} + \mathrm{c.c.}), \quad v_1 = \frac{1}{2}(\mathcal{V}(x)\, e^{inz} + \mathrm{c.c.}), \tag{19}$$

$$\phi_0 = \frac{1}{2}(\mathcal{F}(x)\, e^{inz} + \mathrm{c.c.}), \quad b_1 = \frac{1}{2}(\mathcal{B}(x)\, e^{inz} + \mathrm{c.c.}),$$

we find

$$V'(x) = \frac{C_1 - \frac{1}{2}\beta|\Psi|^2}{\widehat{\nu} + \frac{1}{2}\alpha|\Psi|^2}, \qquad B'(x) = \frac{\widehat{\eta}C_2}{\widehat{\eta}^2 + \frac{1}{2}|\Psi|^2}. \tag{20}$$

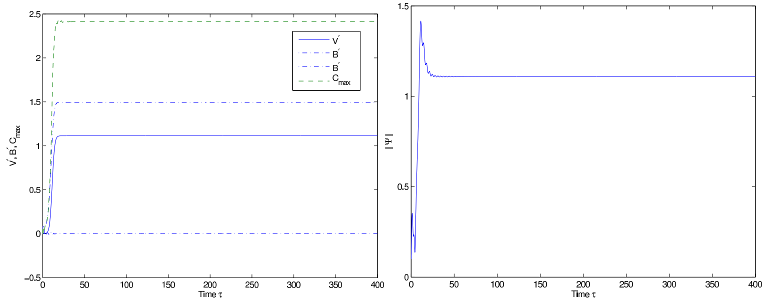
$$\alpha = \frac{\widehat{\nu}v_A^2 + \widehat{\eta}^3 n^2}{\widehat{\eta}^2(v_A^2 + \widehat{\nu}\widehat{\eta}n^2)}, \qquad \beta = \frac{(2\widehat{\Omega} + \widehat{\sigma})\widehat{\eta}^3 n^2 + v_A^2(\widehat{\sigma}\widehat{\nu} - 2\widehat{\Omega}\widehat{\eta})}{\widehat{\eta}^2(v_A^2 + \widehat{\nu}\widehat{\eta}n^2)}. \tag{21}$$

- MRI requires $C_1 = 0$ for nonzero V' and Ψ
- Nonlinear dispersion relation then gives the saturated value of $|\Psi|$:

$$|\Psi|^2 = -\frac{2\widehat{\nu}\widehat{\eta}^2\left[n^2(v_A^2 + \widehat{\nu}\widehat{\eta}n^2)^2 + 2\widehat{\Omega}\widehat{\sigma}v_A^2 + 2\widehat{\Omega}(2\widehat{\Omega} + \widehat{\sigma})\widehat{\eta}^2 n^2\right]}{\left[4\widehat{\Omega}^2 v_A^2\widehat{\eta} + n^2\left(v_A^2 + \widehat{\nu}\widehat{\eta}n^2\right)\left(\widehat{\nu}v_A^2 + \widehat{\eta}^3 n^2\right)\right]} \tag{22}$$

This bifurcation equation determines the saturation amplitude

Approach to Saturated State



- Time-dependent evolution of an x-invariant single-mode perturbation indicates approach to predicted stationary solution
- Above results display extreme cases: disks supported entirely by mechanical ($B' = 0$) or magnetic ($B' \neq 0$) pressure
- $\nu_t = 2\pi\epsilon|\Psi| \sim O(\epsilon)$: turbulent viscosity associated with developed MRI

Small radial scales

If we suppose that

$$(\nu,\eta)=\epsilon(\widehat{\nu},\widehat{\eta}),\quad (\Omega,\sigma)=\delta^{-1}(\widehat{\Omega},\widehat{\sigma}),\quad (n,\lambda)=\delta^{-1}(\widehat{n},\widehat{\lambda}), \tag{23}$$

where $\epsilon \ll 1$, $\delta \ll 1$ with $\epsilon = o(\delta)$ we obtain

$$\tilde{\nabla}^2\psi'_{0t}+2\widehat{\Omega}v'_{1z}=v_A^2\tilde{\nabla}^2\phi'_{0z} \tag{24}$$

$$v_{1t}-(2\widehat{\Omega}+\widehat{\sigma}+V'(x))\psi_{0z}=v_A^2 b_{1z}-v_A^2 B'(x)\phi_{0z} \tag{25}$$

$$\phi_{0t}=\psi_{0z} \tag{26}$$

$$b_{1t}-B'(x)\psi_{0z}=v_{1z}-(\widehat{\sigma}+V'(x))\phi_{0z} \tag{27}$$

$$\widehat{\nu}V'(x)=\overline{\psi_0 v_{1z}}-v_A^2\overline{\phi_0 b_{1z}} \tag{28}$$

$$\widehat{\eta}B'(x)=\overline{\psi_0 b_{1z}}-\overline{\phi_0 v_{1z}} \tag{29}$$

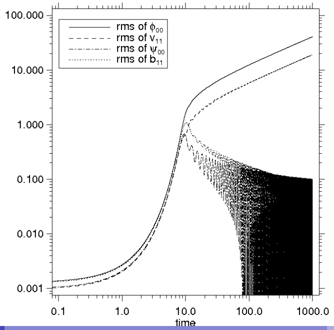
In these equations $\tilde{\nabla}\equiv(\partial_{\tilde{x}},0,\partial_z)$, where $\tilde{x}\equiv x/\delta$ is a fast scale. Hence full spatial dependence is retained but dissipation is subdominant.

Single channel mode

With $\partial_X \overline{v}_0 = \partial_X \overline{b}_0 \equiv 0$ the reduced equations admit exponentially growing solutions of the form (Goodman and Xu, 1994)

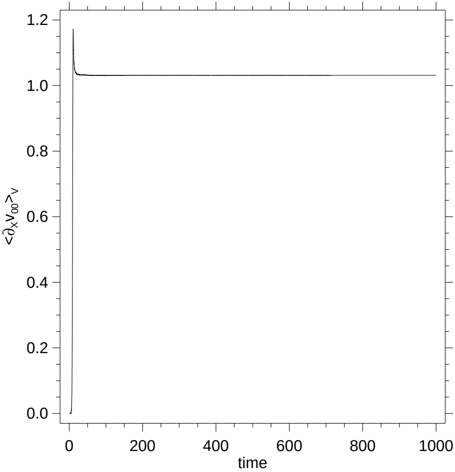
$$\begin{aligned} \psi_0 &= \Psi_0(t)\cos \widehat{n}z, & v_1 &= V_0(t)\sin \widehat{n}z, \\ \phi_0 &= \Phi_0(t)\sin \widehat{n}z, & b_1 &= B_0(t)\cos \widehat{n}z, \end{aligned} \tag{30}$$

However, within the theory an initial state with $\widehat{n} = \widehat{n}_{\max}$ and $\partial_X \overline{v}_0 = \partial_X \overline{b}_0 = 0$ develops nonzero $\partial_X \overline{v}_0$, $\partial_X \overline{b}_0$, resulting in a transition from exponential growth to algebraic growth in time.

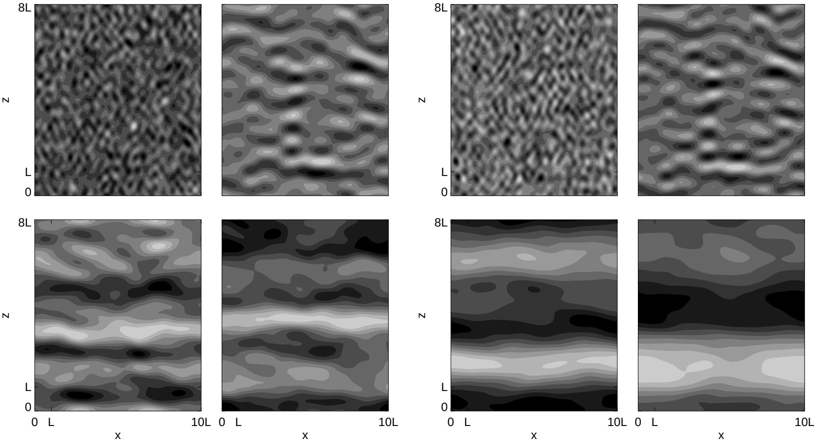


Single channel mode

Despite unbounded algebraic growth and decay in the single channel mode $\langle \partial_X v_0 \rangle_V \to \partial_X \overline{v}_0$ as $t \to \infty$. Thus $\langle \partial_X v_0 \rangle_V$ reaches a **stable** saturated state, as does the transport of angular momentum.

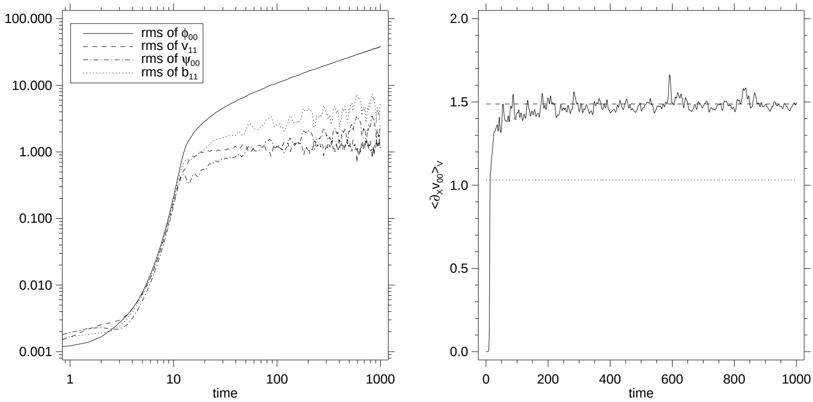


Multiple channel modes



Evolution of (a) $\psi_0/\langle\psi_0^2\rangle_V^{1/2}$ and (b) $\phi_0/\langle\phi_0^2\rangle_V^{1/2}$ at $t = 0, 10, 350, 1000$.

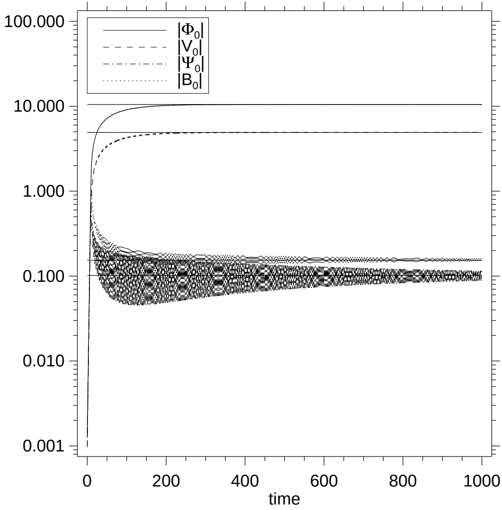
Multiple channel modes



Evolution of an x-dependent initial multiple mode state. The equilibrium values of $\langle \partial_X \overline{v}_0 \rangle_V$ with $\widehat{n}_{\max}$ (dotted) and the smallest vertical wavenumber permitted $\widehat{n}_{\text{eff}}$ (dashed) are also shown.

Subdominant Dissipation

When explicit (ohmic) dissipation $\epsilon_\eta \tilde{\nabla}^2$ is retained (with $\epsilon_\eta = 0.01$) the algebraic growth of the fluctuations also saturates



Theory

When the nonlinear terms $\partial_X \overline{v}_0$, $\partial_X \overline{b}_0$ are ignored the solution of the reduced equations is

$$(\Psi_0(t), V_0(t), \Phi_0(t), B_0(t)) \equiv \left(1, -2\frac{\widehat{n}_{\max}}{\widehat{\sigma}}, -\widehat{n}_{\max}, 2\frac{\widehat{n}^2_{\max}}{\widehat{\sigma}}\right) e^{\lambda_{\max} t}.$$

This solution is in fact an exact solution of the nonlinear fluctuating equations as obtained by Goodman and Xu (1994). But when $\partial_X \overline{v}_0$, $\partial_X \overline{b}_0 = 0$ are included the exponential growth becomes algebraic:

$$\begin{aligned} \psi_0 &= (\Psi_1 t^{-1/2} + \Psi_2 \cos\omega t)\cos(\widehat{n}z) \\ v_1 &= (V_1 t^{1/2} + V_2 \sin\omega t)\sin(\widehat{n}z) \\ \phi_0 &= (\Phi_1 t^{1/2} + \Phi_2 \sin\omega t)\sin(\widehat{n}z) \\ b_1 &= (B_1 t^{-1/2} + B_2 \cos\omega t)\cos(\widehat{n}z), \end{aligned} \tag{31}$$

provided $\partial_X \overline{v}_0 = -\widehat{\sigma} - \frac{v_A^2 \widehat{n}^2}{2\widehat{\Omega}}$ and $\omega = \sqrt{4\widehat{\Omega}^2 + \widehat{n}^2 v_A^2}$.

Theory

From the closure relation one obtains

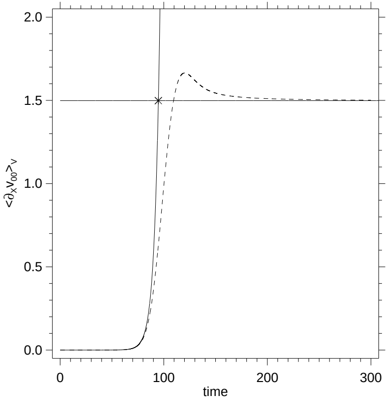
$$\widehat{\nu} \left\langle \partial_X v_0 \right\rangle_V = \frac{\widehat{n}^2\omega^2}{2\widehat{\Omega}}\Psi_1^2 - \frac{\widehat{n}^2\omega}{8\widehat{\Omega}} \sin 2\omega t \Psi_2^2. \tag{32}$$

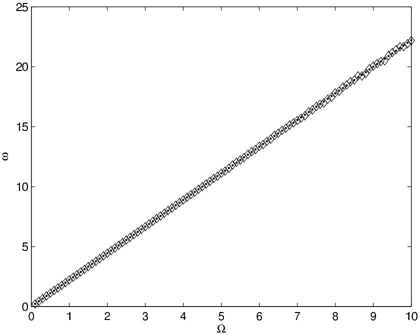
It is remarkable that this expression does not contain secular terms proportional to $t^{1/2}\cos\omega t$, $t^{-1/2}\sin\omega t$ or indeed t, and hence saturates despite the algebraic growth of the contributing fields (cf. Landau damping). The mean component arises from products of the terms $\Phi_1 t^{\frac{1}{2}}, V_1 t^{\frac{1}{2}}$ and $\Psi_1 t^{-\frac{1}{2}}, B_1 t^{-\frac{1}{2}}$, while the oscillatory component is a consequence of the terms $(\Psi_2, V_2, \Phi_2, B_2)$. On time-averaging (32) we obtain finally the prediction

$$\Psi_1^2 = \frac{2\widehat{\nu}\widehat{\Omega}}{\widehat{n}^2\omega^2}\partial_X \overline{v}_0 = \frac{\widehat{\nu} v_A^2}{\widehat{n}^2\omega^2}\left(-\frac{2\widehat{\Omega}\widehat{\sigma}}{v_A^2} - \widehat{n}^2\right). \tag{33}$$

Theory

We can measure the frequency ω from the numerical simulations for a range of values of $\widehat{\Omega}$, with the remaining parameters fixed.





(a) Back-reaction saturates the growth of $\langle \partial_X v_0 \rangle_V$, (b) $\omega(\widehat{\Omega})$

Summary

- Simple scaling suffices to characterize a one-parameter family of self-consistent equilibrated states
  - Strong modification of the background shear that feeds the MRI
  - Equilibration ultimately determined by ohmic + viscous dissipation
  - This regime is not accessible to fully resolved simulations

Details in:

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